Solutions and laws of conservation for coupled nonlinear Schrödinger equations: Lie group analysis

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A set of two coupled nonlinear Schrödinger equations is systematically analyzed by means of Lie group technique. The physical situations under consideration include nonlinear propagation in strongly birefringent and multimode optical fibers. The most general Lie group of point symmetries, its Lie algebra, and a group of adjoint representations that correspond to the Lie algebra are identified. As a result, a complete list of group-invariant exact solutions is obtained and compared with earlier results. The corresponding laws of conservation are derived employing Noether's theorem. [S1063-651X(98)10501-9]

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I. INTRODUCTION

A set of coupled nonlinear Schrödinger equations (CNSEs) is a basic mathematical model in different branches of physics. Very often, CNSEs appear in nonlinear fiber optics [1,2], where their different versions describe nonlinear pulse propagation in, e.g., multimode optical fibers [3], birefringent fibers [4,5], couplers [6–8], four-wave mixing [9–11], and Raman scattering [12,13]. The present study focuses on the CNSEs ($\nu=\pm1$)

$$iA_x+\frac{1}{2}A_{tt}+(|A|^2+h|B|^2)A=0,$$
$$iB_x+\frac{\nu}{2}B_{tt}+(|B|^2+h|A|^2)B=0. \tag{1}$$

They describe a propagation of (a) two waves at different carrier wavelengths in two-mode optical fibers ($h=2$) [3] and (b) two modes in fibers with strong birefringence ($h=\frac{2}{3}$) [2]. In both cases x and t denote the normalized distance and time. A and B are normalized slowly varying amplitudes of waves with a different carrier wavelength, or the polarization components of the wave. Note that in order to get Eqs. (1) in the case of strong birefringence, an additional change of dependent variables has been made [2], after which the terms that describe the effect of separation (walk-off effect [2]) between the two polarization components disappear. $\nu=1$ ($\nu=-1$) describes the propagation in the region of negative (positive) group-velocity dispersion (GVD).

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As it is well known, for $h=1$ and $\nu=1$ Eqs. (1) are integrable by means of the inverse scattering method [14]. Recently, N-soliton solutions also have been obtained for this integrable case by the Hirota method [15]. For $\nu=-1$ and a large enough cross-phase modulation (CPM) ($h>1$) ''bright''(''dark'') solitons and the corresponding more general periodic solutions can exist in the region with positive (negative) GVD [16,17,11]. The physical effect responsible for such possibility is the CPM. Such periodic waves and solitons are called symbiotic [17,11]. For arbitrary values of h and ν, however, these equations are no longer integrable [18]. Numerical studies of Eqs. (1) have been reviewed in [1,2]. A systematic investigation of exact solutions of Eqs. (1) for $\nu=1$ has been performed by the similarity method in [19,20].

At the same time, it is well known that Lie group analysis is one of the feasible ways of providing a possibility for various exact solutions or classes of exact solutions to be specified. The crucial idea of the Lie group method is based on the natural symmetries possessed by any system of partial differential equations. Using a well-known procedure [21,22], a certain number of reduced ordinary differential equations can be obtained. Their solutions constitute an optimal set of group-invariant solutions. This means that a special kind of group classification of one sort of solutions to the original system of partial differential equations can be made. The Lie group method was successfully used to produce exact solutions for the higher-order Schrödinger equation [23] as well as for a pair of linearly coupled CNSEs [24].

The purpose of this paper is to present a Lie group-classification of one-parameter group-invariant solutions of Eqs. (1). The classification obtained and the exact solutions for $\nu=1$ are compared with earlier results of [19]. Note however, that in addition to [19], we study the propagation of two waves in different GVD regions, i.e., $\nu=-1$. This makes an analysis of the symbiotic periodic waves (SPWs) and solitons possible.

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In Sec. II we briefly describe the method and simultaneously present our main results obtained from its application: (i) the most general Lie group of point symmetries and its Lie algebra and (ii) the group of adjoint representations of the Lie algebra.

In Sec. III we proceed with the Lie group classification giving the optimal system of one-dimensional subalgebras and the corresponding optimal set of reduced systems. There are six reduced systems (cases A and B) consisting of first-order differential equations that are readily integrated into uniform wavetrains (case A) and vanishing waves with nontrivial simultaneous dependence of phase on t and x (case B). The rest of the systems are of second order. Different periodic (and in particular cases soliton) solutions, including the symbiotic ones for $\nu=-1$, are obtained.

In Sec. IV we use Noether's theorem to obtain conservation laws associated with the Hamiltonian symmetries of Eqs. (1). In Sec. V we discuss the invariant solutions obtained. A comparison is also made with the solutions for $\nu=1$ published in [19,20]. The full forms of the solutions that compose the families of conjugate solutions are given in Appendix A. Appendix B illustrates how the exact solutions found in [19] can be obtained from those derived here.

II. BASIC RESULTS OF THE LIE GROUP ANALYSIS

Throughout the paper for the complex functions $A(t,x)$ and $B(t,x)$ we write either $A=ze^{i\alpha}$, $B=\zeta e^{i\beta}$ or $A=u+iv$, $B=w+is$, where $z,\zeta,\alpha,\beta,u,v,w,s$ are real quantities that depend on t and x. Let us consider the space of the variables $(\chi,\pi,\pi^{(1)},\pi^{(2)})$ in which Eqs. (1) define the differential manifold

$$F(\chi,\pi,\pi^{(1)},\pi^{(2)})=0,$$

denoting by $\chi=\{\chi^i\}_{i=1,2}$ the set of independent variables $\{t,x\}$ and by $\pi=\{\pi^k\}_{k=1,2,3,4}$ either of the two sets of dependent variables $\{z,\zeta,\alpha,\beta\}$ or $\{u,v,w,s\}$; $\pi^{(1)}=\{\pi_i^k\}_{i=1,2}^{k=1,2,3,4}$ and $\pi^{(2)}=\{\pi_{i,j}^k\}_{i,j=1,2}^{k=1,2,3,4}$ are substituted for the partial derivatives of first and second order, respectively, and $F=(F_1,F_2,F_3,F_4)$ denotes the left-hand sides of Eqs. (1), expressed in terms of real variables. The infinitesimal criterion under which Eqs. (1) are invariant in regard to the group G of point transformations

$$\begin{aligned}\chi'^i&=f^i(\chi,\pi,a),\quad f^i_{|a=0}=\chi^i,\quad i=1,2\\ \pi'^k&=\varphi^k(\chi,\pi,a),\quad \varphi^k_{|a=0}=\pi^k,\quad k=1,2,3,4\end{aligned}\tag{2}$$

$(a\in\Delta\subset R,\ 0\in\Delta)$ consists of a linear homogeneous system of equations for the coordinates $\xi^i(\chi,\pi),\eta^k(\chi,\pi)$ of the infinitesimal generator $X=\xi^i(\chi,\pi)(\partial/\partial\chi^i)+\eta^k(\chi,\pi)(\partial/\partial\pi^k)$

$$p^{(2)}X(F)_{|F=0}=0,\tag{3}$$

where $p^{(2)}X$ denotes the second prolongation of the operator X with respect to the derivatives $\pi^{(1)}$ and $\pi^{(2)}$,

$$p^{(2)}X=X+\zeta_i^k\frac{\partial}{\partial\pi_i^k}+\zeta_{ij}^k\frac{\partial}{\partial\pi_{ij}^k}.$$

A sum must be taken over the duplicate indices. The coefficients ζ_i^k and ζ_{ij}^k depend on the functions $\xi^i(\chi,\pi),\eta^k(\chi,\pi)$ and their derivatives [21,22]. For the group G the system (3) is the so-called defining system of equations. Its full set of solutions constitutes a Lie algebra that generates the widest permissible local group of continuous point transformations for the system of differential equations under consideration.

Since all the variables $(\chi,\pi,\pi^{(1)},\pi^{(2)})$ are independent, the defining system is over-determined, which facilitates the solution. However, the great number of 135 equations requires an essential use of a certain language for machine computing. By utilizing the package of the computer system for symbolic calculations MATHEMATICA [25], we have written several specific programming modules for solving some distinctive types of linear partial differential equations with constant coefficients. Without having been done any prior assignments to h, by rerunning the modules repeatedly, we obtained the solution to Eq. (3): a six-dimensional Lie algebra with the basis of generators

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=\frac{\partial}{\partial\alpha},\quad X_4=\frac{\partial}{\partial\beta},$$

$$X_5=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial\alpha}+\nu t\frac{\partial}{\partial\beta},\tag{4}$$

$$X_6=-t\frac{\partial}{\partial t}-2x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}+\zeta\frac{\partial}{\partial\zeta}.$$

Furthermore, we implemented the procedure of calculation, preliminarily specifying for the parameter h physically relevant values $h=1,2,\frac{2}{3}$. The result shows the existence of two additional infinitesimal operators only for $\nu=1$ and $h=1$:

$$X_7=\zeta\,\cos(\beta-\alpha)\frac{\partial}{\partial z}-z\,\cos(\beta-\alpha)\frac{\partial}{\partial\zeta}+\frac{\zeta}{z}\sin(\beta-\alpha)\frac{\partial}{\partial\alpha}+\frac{z}{\zeta}\sin(\beta-\alpha)\frac{\partial}{\partial\beta},$$

$$X_8=\zeta\,\sin(\beta-\alpha)\frac{\partial}{\partial z}-z\,\sin(\beta-\alpha)\frac{\partial}{\partial\zeta}-\frac{\zeta}{z}\cos(\beta-\alpha)\frac{\partial}{\partial\alpha}-\frac{z}{\zeta}\cos(\beta-\alpha)\frac{\partial}{\partial\beta}.$$

This is noteworthy in view of the fact that in the region of negative GVD for both modes and $h=1$ the system (1) has an infinite set of constants of motion and may be solved by the inverse scattering transform [14], whereas for $h\neq1$ the system is found to be nonintegrable by inverse scattering.

From now on in this paper we shall primarily deal with the nonintegrable case, but it should be noted that the reduced equations apply to arbitrary h. Nevertheless, one must remember that the group investigation, presented here, is completely applicable and exhaustive in the framework of Lie theory only for those values of h for which the widest permissible Lie algebra of the related Eqs. (1) coincides with

TABLE I. Commutators of the basis vectors of Lie algebra.

X_i	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	$X_3+\nu X_4$	$-X_1$
X_2	0	0	0	0	X_1	$-2X_2$
X_3	0	0	0	0	0	0
X_4	0	0	0	0	0	0
X_5	$-X_3-\nu X_4$	$-X_1$	0	0	0	X_5
X_6	X_1	$2X_2$	0	0	$-X_5$	0

the algebra based on the vectors (4). The commutators $[X_i,X_j]$ of the basis vector fields (4) are shown in the Table I.

Using the Lie equation

$$\frac{df^i}{da}=\xi^i(f,\varphi),\quad f^i_{|a=0}=\chi^i,\quad i=1,2$$

$$\frac{d\varphi^k}{da}=\eta^k(f,\varphi),\quad \varphi^k_{|a=0}=\pi^k,\quad k=1,2,3,4$$

$(f=\{f^i\}_{i=1,2},\ \varphi=\{\varphi^k\}_{k=1,2,3,4})$, we obtain six families of one-parameter transformations (2) in conformity with the infinitesimal operators (4): (i) time translation T_{a_1}, with $t'=t+a_1$; (ii) space translation T_{a_2}, with $x'=x+a_2$; (iii) rotation of the phase α, T_{a_3}, with $\alpha'=\alpha+a_3$; (iv) rotation of the phase β, T_{a_4}, with $\beta'=\beta+a_4$, (v) Galilean boost and simultaneous phase transformations T_{a_5}, with $t'=t+a_5x$, $\alpha'=\alpha+a_5t+(a_5^2/2)\,x$, and $\beta'=\beta+\nu a_5t+\nu\,(a_5^2/2)\,x$; and (vi) heterogeneous scaling of time, space, and amplitudes T_{a_6}, with $t'=e^{-a_6}t$, $x'=e^{-2a_6}x$, $z'=e^{a_6}z$, and $\zeta'=e^{a_6}\zeta$. The most general symmetry group G of Eqs. (1) (in the nonintegrable case, as indicated above) is a six-parameter transformation T_a, with

$$t'=e^{-a_6}t+e^{-a_6}a_5x+a_1, \tag{5}$$

$$x'=e^{-2a_6}x+a_2,$$

$$z'=e^{a_6}z,$$

$$\zeta'=e^{a_6}\zeta,$$

$$\alpha'=\alpha+a_5t+\frac{a_5^2}{2}x+a_3,$$

$$\beta'=\beta+\nu a_5t+\nu\frac{a_5^2}{2}x+a_4,$$

with a vector-parameter $a=(a_1,\ldots,a_6)$.

Generated by the basis vectors (4), there exist six adjoint representations (interior automorphisms) $A_i(\varepsilon)$ $(i=1,\ldots,6;\ \varepsilon\in R)$ of the Lie algebra [21,22], acting on X_j according to the Table II. The general automorphism A is given by the composition

$$A(\varepsilon_1,\varepsilon_2,\varepsilon_5,\varepsilon_6)=A_1(\varepsilon_1)\circ A_2(\varepsilon_2)\circ A_5(\varepsilon_5)\circ A_6(\varepsilon_6), \tag{6}$$

where $A_i(\varepsilon_i)\circ A_j(\varepsilon_j)X=A_i(\varepsilon_i)\,(A_j(\varepsilon_j))X$. The main results obtained in this section, the permissible group of symmetries G (5) and the group of interior automorphisms (6) of the associated Lie algebra, are applied in the next section to perform a full classification of one-parameter group-invariant solutions of Eqs. (1).

III. OPTIMAL SET OF LIE ALGEBRAS AND THE CORRESPONDING REDUCED SYSTEMS: INVARIANT SOLUTIONS

There is an infinite number of subgroups of the general group of symmetries G useful for yielding special exact solutions or classes of exact solutions that are invariant under at least one of the subgroups. However, a well-known standard procedure [21,22] makes it possible to classify all the invariant solutions in subsets of conjugate solutions. The adjoint representations (6) introduce a conjugate relation in the set of all subalgebras of the same dimension. If we take only one representative from each family of equivalent subalgebras, an optimal set of subalgebras is created. For the system under consideration we built up the optimal set consisting of one-dimensional not conjugate subalgebras, which we present in a compact form of six unified cases: case A,

$$X_1+\varepsilon X_3=\frac{\partial}{\partial t}+\varepsilon\,\frac{\partial}{\partial\alpha}\quad(\varepsilon=0,\pm1);$$

case B,

$$\varepsilon X_4+X_5=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial\alpha}+(\varepsilon+\nu t)\frac{\partial}{\partial\beta}\quad(\varepsilon=0,\pm1);$$

case C,

TABLE II. Interior automorphisms generated by the basis vectors of the Lie algebra.

$A_i(\epsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_1(\varepsilon)$	X_1	X_2	X_3	X_4	$X_5-\varepsilon(X_3+\nu X_4)$	$X_6+\varepsilon X_1$
$A_2(\varepsilon)$	X_1	X_2	X_3	X_4	$X_5-\varepsilon X_1$	$X_6+2\varepsilon X_2$
$A_3(\varepsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_4(\varepsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_5(\varepsilon)$	$X_1+\varepsilon(X_3+\nu X_4)$	$X_2+\varepsilon X_1+\frac{\varepsilon^2}{2}(X_3+\nu X_4)$	X_3	X_4	X_5	$X_6-\varepsilon X_5$
$A_6(\varepsilon)$	$e^{-\varepsilon}X_1$	$e^{-2\varepsilon}X_2$	X_3	X_4	$e^{\varepsilon}X_5$	X_6

$$X_2+\delta X_3+\varepsilon X_4=\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial\alpha}+\varepsilon\frac{\partial}{\partial\beta}$$

$$(\varepsilon=0,\ \delta=0,\pm1\quad\text{or}\quad\varepsilon=\pm1,\delta\in R);$$

case D,

$$\varepsilon X_2+\delta X_4+X_5=x\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial x}+t\frac{\partial}{\partial\alpha}+(\delta+\nu t)\frac{\partial}{\partial\beta}$$

$$(\varepsilon=\pm1,\ \delta\in R);$$

case E,

$$\varepsilon X_3+\delta X_4+X_6=-t\frac{\partial}{\partial t}-2x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}+\zeta\frac{\partial}{\partial\zeta}+\varepsilon\frac{\partial}{\partial\alpha}+\delta\frac{\partial}{\partial\beta}\quad(\varepsilon,\delta\in R);$$

and case F,

$$\varepsilon X_3+\delta X_4=\varepsilon\frac{\partial}{\partial\alpha}+\delta\frac{\partial}{\partial\beta}\quad(\varepsilon=1,\ \delta=0\quad\text{or}\ \varepsilon\in R,\delta=1).$$

By setting various possible values for the parameters ε and δ one obtains different elements of the optimal set.

According to the main assertion of the group theory the invariant solutions are obtained from special reduced systems of equations that are derived from the original system of partial differential equations. For this purpose functionally independent quantities that are invariant under the group transformations are substituted for both independent and dependent variables. We applied the scheme of reduction to each one of the subgroups from case A to case E (there are no invariant solutions for the case F) and obtained the optimal set of reduced systems of ordinary differential equations (7), (9), (12), (21), and (23). Without citing details, we are going to present the reduced systems and subsequently discuss some of their solutions. Throughout this section prime denotes differentiation.

Case A. After the substitutions $z=p(x)$, $\zeta=q(x)$, $\alpha=f(x)+\varepsilon t$, and $\beta=g(x)$ are made Eqs. (1) reduce to

$$\begin{gathered}p'=q'=0,\\ f'=p^2+hq^2-\frac{\varepsilon^2}{2},\\ g'=q^2+hp^2.\end{gathered}\tag{7}$$

The general solution of Eqs. (7) readily shows that the invariant solutions of Eqs. (1) in this case are uniform wave trains ($C_1\geqslant0$, $C_2\geqslant0$, and $\varepsilon=0,\pm1$)

$$\begin{gathered}A=C_1\exp i\left\{\left(C_1^2+hC_2^2-\frac{\varepsilon^2}{2}\right)x+\varepsilon t\right\},\\ B=C_2\exp i\{(C_2^2+hC_1^2)x\}.\end{gathered}\tag{8}$$

Case B. The substitutions $z=p(x)$, $\zeta=q(x)$, $\alpha=f(x)+t^2/2x$, and $\beta=g(x)+\nu(t^2/2x)+\varepsilon(t/x)$ lead to a reduced system of equations

$$\begin{gathered}2xp'+p=0,\\ 2xq'+q=0,\\ f'=p^2+hq^2,\\ g'=q^2+hp^2-\nu\frac{\varepsilon^2}{2x^2}.\end{gathered}\tag{9}$$

Its general solution allows invariant solutions that decay in amplitude as x increases with a nontrivial dependence of phase on t and x,

$$\begin{gathered}A=\sqrt{\frac{C_1}{x}}\exp i\left\{(C_1+hC_2)\ln|x|+\frac{t^2}{2x}\right\},\\ B=\sqrt{\frac{C_2}{x}}\exp i\left\{(C_2+hC_1)\ln|x|+\nu\frac{(t+\nu\varepsilon)^2}{2x}\right\},\end{gathered}\tag{10}$$

where the real constants C_1 and C_2 have the same sign as the variable $x(\varepsilon=0,\pm1)$.

Case C. The invariant solutions are of the form

$$\begin{gathered}A=p(t)\exp i\{f(t)+\delta x\},\\ B=q(t)\exp i\{g(t)+\varepsilon x\}.\end{gathered}\tag{11}$$

After inserting these expressions into Eqs. (1) we obtain the reduced system

$$\begin{gathered}2p'f'+pf''=0,\\ 2q'g'+qg''=0,\\ p''-p(f')^2+2p^3+2hpq^2-2\delta p=0,\\ q''-q(g')^2+\nu2q^3+\nu2hqp^2-\nu2\varepsilon q=0.\end{gathered}\tag{12}$$

It clearly has a set of solutions of uniform wave trains for ($\varepsilon=0$, $\delta=0,\pm1$ or $\varepsilon=\pm1$, $\delta\in R$)

$$\begin{gathered}A=C_1\exp i\{\sigma\sqrt{2(C_1^2+hC_2^2-\delta)}t+\delta x\},\\ B=C_2\exp i\{\sigma\sqrt{\nu2(C_2^2+hC_1^2-\varepsilon)}t+\varepsilon x\},\end{gathered}\tag{13}$$

where $\sigma=\pm1$, $C_1\geqslant0$, and $C_2\geqslant0$, which are similar but not conjugate to the solutions (8).

Here we exhibit separately some particular solutions of (12) for $\nu=1$ and $\nu=-1$.

$\nu=1$. Requiring $\delta=\varepsilon$, $q=p$, and $g=\pm f+\text{const}$, the system (12) takes the form

$$\begin{gathered}p^2f'=C_1,\\ p''+2(h+1)p^3-2\varepsilon p-\frac{C_1^2}{p^3}=0\end{gathered}$$

(C_1 is a real constant). The solution for this system are knoidal waves with a phase expressed by the third-order elliptic integral $\Pi(n;j|m)=\int_0^j[1-n\mathrm{sn}^2(w|m)]^{-1}\,dw$ [26]. The explicit form of the solution is

$$A=U\exp i\left\{\frac{C_1}{2\lambda\sqrt{h+1}b_1}\Pi(n;j|m)+\varepsilon x\right\},$$

$$B=U\exp i\left\{\frac{\pm C_1}{2\lambda\sqrt{h+1}b_1}\Pi(n;j|m)+\varepsilon x\right\},$$

$$U=\sqrt{(b_1-b_2)\mathrm{cn}^2\,(j|m)+b_2}, \tag{14}$$

$$j=2\lambda\sqrt{h+1}t,\quad \lambda=\frac{1}{2}\sqrt{b_1-b_3},$$

$$m=\frac{b_1-b_2}{b_1-b_3},\ n=\frac{b_1-b_2}{b_1},\quad \varepsilon=0,\pm 1,$$

where $b_1>b_2>b_3$ are the roots of the polynomial

$$Q(\theta)=\theta^3-\frac{2\varepsilon}{h+1}\theta^2-\frac{C_2}{4(h+1)}\theta+\frac{C_1^2}{h+1},$$

C_2 is a real constant, and $\mathrm{cn}(j|m)$ denotes the Jacobian cosine elliptic function with parameter m.

If we assume $\varepsilon=1$ and either $b_3=0$, $b_2\to 0$, or $b_2=0$, $b_3\to 0$, then we obtain from Eqs. (14) the soliton solution

$$A=B=\sqrt{\frac{2}{h+1}}\,\mathrm{sech}\,(\sqrt{2}t)\exp(ix). \tag{15}$$

$\nu=-1$. Assume $g'=f'=0$. After the substitution of the ansatz $p=C_1\mathrm{dn}(jt|m)$, $q=C_2\mathrm{sn}(jt|m)$ for normal periodic waves (NPWs) and the ansatz $p=C_1\mathrm{sn}(jt|m)$, $q=C_2\mathrm{dn}(jt|m)$ for SPWs in Eqs. (12) we obtained the following invariant solutions. For NPWs (possible for $h<1$),

$$A=\sqrt{\frac{2\varepsilon}{m(1-h)+h+1}}\,\mathrm{dn}\,(jt|m)\exp(i\delta x),$$

$$B=\sqrt{\frac{2\varepsilon m}{m(1-h)+h+1}}\,\mathrm{sn}(jt|m)\exp(i\varepsilon x), \tag{16}$$

$$j=\sqrt{\frac{2\varepsilon(1-h)}{m(1-h)+h+1}},$$

$$m(1-h)(\varepsilon+\delta)=2\varepsilon-\delta h-\delta,\quad \varepsilon=\pm 1,\quad \delta\in R.$$

In order to obtain physically admissible solutions for optical fibers ($0<h<1$) one must take $\varepsilon=1$ and $\delta\in[(h+1)/2,2/(h+1)]$, while $\varepsilon=-1$ is acceptable only for $h<-1$. For SPWs (existing only for $h>1$),

$$A=\sqrt{\frac{2(h+1-2\delta)}{(h-1)(h+3)}}\,\mathrm{sn}(jt|m)\exp(i\delta x),$$

$$B=\sqrt{\frac{2(\delta+1)}{h+3}}\,\mathrm{dn}(jt|m)\exp(ix), \tag{17}$$

$$j=\sqrt{\frac{2(h-1)(\delta+1)}{h+3}},\quad m=\frac{h+1-2\delta}{(h-1)(\delta+1)},$$

$$\frac{2}{h+1}<\delta<\frac{h+1}{2}.$$

If $m\to 1$, then Eqs. (16) and (17) converge to the corresponding normal and symbiotic solitons: for normal solitons ($h<1$),

$$A=\mathrm{sech}(\sqrt{1-h}t)\exp\left(i\,\frac{(h+1)x}{2}\right),$$

$$B=\tanh(\sqrt{1-h}\,t)\exp(ix), \tag{18}$$

and for symbiotic solitons ($h>1$),

$$A=\sqrt{\frac{2}{h+1}}\tanh\left(\sqrt{\frac{2(h-1)}{h+1}}\,t\right)\exp\left(i\,\frac{2x}{h+1}\right),$$

$$B=\sqrt{\frac{2}{h+1}}\,\mathrm{sech}\left(\sqrt{\frac{2(h-1)}{h+1}}\,t\right)\exp(ix). \tag{19}$$

The restrictive conditions for h, as outlined above, are quite distinctive for each of the previous waves. They closely correspond to the physical situation and may serve to distinguish which of the two phenomena, self-phase or cross-phase modulation, has a stronger influence on the coupling of the modes (compare with [16,17,11]).

An additional reciprocal transformation of the parameter m [26] in Eqs. (16) and (17) brings forth solutions expressed by the combinations (cn,sn) and (sn,cn), some of which are given elsewhere [27,28]. NPWs (SPWs) and normal (symbiotic) solitons are possible for $\nu=1$ and $h=1$ as well.

Case D. The invariant solutions are given by the expressions

$$A=p(y)\exp i\left\{f(y)+\varepsilon tx-\frac{x^3}{3}\right\},$$

$$B=q(y)\exp i\left\{g(y)+\varepsilon(\delta+\nu t)x-\nu\frac{x^3}{3}\right\}, \tag{20}$$

$$y=t-\varepsilon\,\frac{x^2}{2},$$

where the functions $p(y)$, $q(y)$, $f(y)$, and $g(y)$ are solutions to the reduced system of equations

$$2p'f'+pf''=0,$$

$$2q'g'+qg''=0,$$

$$p''-p(f')^2+2p^3+2hpq^2-2\varepsilon yp=0, \tag{21}$$

$$q''-q(g')^2+\nu 2q^3+\nu 2hqp^2-2\varepsilon yq-\nu 2\varepsilon\delta q=0$$

($\varepsilon=\pm 1$, $\delta\in R$). The first two of these equations may be integrated as

$$p^2f'=C_1,\quad q^2g'=C_2\quad (C_1,C_2=\mathrm{const}).$$

Considering $\nu=1$, we assume $\delta=0$, $q=p$, and $g=\pm f+\text{const}$. Then the last two equations of Eqs. (21) are transformed to

$$p''+2(h+1)p^3-2\varepsilon yp-\frac{C_1^2}{p^3}=0,\quad \varepsilon=\pm 1.$$

Case E. We complete the optimal set of one-parameter group-invariant solutions, presenting here the last invariant solutions

$$A=\frac{p(y)}{t}\exp\, i\{f(y)-\varepsilon\,\ln|t|\},$$

$$B=\frac{q(y)}{t}\exp\, i\{g(y)-\delta\,\ln|t|\},\tag{22}$$

$$y=\frac{t^2}{x}$$

$(\varepsilon,\delta\in R)$, which depend on the solutions of the most complicated system of reduced equations

$$4y^2pf''+8y^2p'f'-2y^2p'-4\varepsilon yp'-2ypf'+3\varepsilon p=0,$$

$$4y^2qg''+8y^2q'g'-\nu 2y^2q'-4\delta yq'-2yqg'+3\delta q=0,$$

$$4y^2p''-4y^2p(f')^2+4\varepsilon ypf'+2y^2pf'-2yp'+2hpq^2+2p^3-\varepsilon^2p+2p=0,$$

$$4y^2q''-4y^2q(g')^2+4\delta yqg'+\nu 2y^2qg'-2yq'+\nu 2hqp^2+\nu 2q^3-\delta^2q+2q=0.\tag{23}$$

Following the previous presentation, it is easy to disjoint the whole set of invariant solutions (8), (10), (11), (13)–(20), and (22) into two optimal sets of one-parameter group-invariant solutions for $\nu=1$ and $\nu=-1$. Each of these optimal sets comprises all of the invariant solutions, subject to the reduced systems of equations (7), (9), (12), (21), and (23) for the respective ν. By acting with the maximal group of symmetry transformations (5), each of the solutions from the optimal set generates a family of conjugate invariant solutions that depend on six additional parameters $a_1,\ldots,a_6$ (see Appendix A).

IV. CONSERVATION LAWS

Now we apply a version of Noether's theorem [21], which provides a very useful tool for obtaining conservation laws that hold for the Hamiltonian type of systems of equations. For that purpose we set $\pi=\{u,v,w,s\}$ and by the use of the Hamiltonian matrix

$$\mathcal{D}=\begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2} & 0\end{pmatrix}$$

we recast the coupled pair of nonlinear Schrödinger equations (1) as a Hamiltonian system

$$\frac{\partial\pi}{\partial x}=\mathcal{D}\delta(\mathcal{H}),$$

with the Hamiltonian functional

$$\mathcal{H}[\pi]=\int_{-\infty}^{\infty}[-\tfrac{1}{2}(|A_t|^2+\nu|B_t|^2)+\tfrac{1}{2}(|A|^4+|B|^4)+h|A|^2|B|^2]dt,$$

denoting by $\delta(\mathcal{L})=\{\delta_u(\mathcal{L}),\delta_v(\mathcal{L}),\delta_w(\mathcal{L}),\delta_s(\mathcal{L})\}$ the variational derivative of a functional $\mathcal{L}[\pi]=\int_{-\infty}^{\infty}L[\pi]dt$. Next we verify whether any of the groups of point symmetries presented in Sec. II are Hamiltonian, namely, we look for such symmetries that possess characteristic $Q=\{Q_k\}_{k=1,2,3,4}$, defined by $Q_k(\chi,\pi,\pi^{(1)})=\eta^k(\chi,\pi)-\xi^i(\chi,\pi)\pi_i^k$, for which there exists a functional $\mathcal{J}[\pi]$ satisfying the condition

$$Q=\mathcal{D}\delta(\mathcal{J}).\tag{24}$$

Considering the nonintegrable case, we found that Eq. (24) holds for all symmetries except one: the symmetry of scaling T_{a_6}. Hence Noether's theorem implies that there exist five independent nontrivial conservation laws $\mathcal{J}_1,\ldots,\mathcal{J}_5$ associated with each of the Hamiltonian symmetries $T_{a_1},\ldots,T_{a_5}$, respectively: (i) the ''momentum'' of the solution (the asterisk means complex conjugate)

$$\mathcal{J}_1=\int_{-\infty}^{\infty}(A_tA^*+B_tB^*)dt,$$

(ii) the Hamiltonian of the system $\mathcal{J}_2\equiv\mathcal{H}$, (iii) the energy of the first mode $\mathcal{J}_3=\int_{-\infty}^{\infty}|A|^2dt$, (iv) the energy of the second mode $\mathcal{J}_4=\int_{-\infty}^{\infty}|B|^2dt$, and (v) the initial ''center of mass'' of the solution

$$\mathcal{J}_5=\int_{-\infty}^{\infty}t(|A|^2+\nu|B|^2)dt+ix\mathcal{J}_1.$$

The conservation of energy in each of the channels has a simple physical meaning: In this system there is only ''reactive'' interaction of pulses, i.e., interaction connected with a transfer of phases but without exchange of energy between the channels (compare with [29]).

V. DISCUSSION

We have obtained three types of exact solutions of Eqs. (1) (see Appendix A): (i) uniform wave trains (A1) and (A4) (ii) vanishing waves with a nontrivial simultaneous dependence of phase on t and x (A2), and (iii) different kinds of periodic waves, including symbiotic ones (A5), (A7), and (A8) [and as particular cases soliton solutions (A6), (A9), and (A10)]. Physically more important are types (ii) and (iii). An interesting circumstance has to be mentioned for Eqs. (A2), which we have found for both $\nu=\pm 1$. Note that for $\nu=1$ these solutions were obtained in [19]. Equations (A2) can be interpreted as a kind of static decaying radiative solution [24]. Recall that the solution of NLSEs for nonsoliton initial conditions transforms into a soliton and small radiative parts that are usually thought to be a perturbation. The amplitude and phase dependences on x and t of decaying radiative parts are very similar to those in Eqs. (A2). The small differences are due to the fact that Eqs. (A2) are exact solutions, while the form of decaying radiative parts is a consequence of a certain asympthotic expansion. As Eqs. (A2) are exact solutions of Eqs. (1), it follows that there is no need for them to be considered as a perturbation.

Concerning different kinds of obtained periodic waves (and soliton solutions), we would like to mention two circumstances. First, for $\nu=1$ we have found an exact solution (A5), which has the form of periodic waves with a rather involved phase dependence on t and x given by the third-order elliptic integral. Its particular case is the set of soliton solutions (A6). Second, we have shown that (for $\nu=-1$) the symbiotic periodic waves (A8) and solitons (A10) are invariant solutions of case C type.

Exact solutions of Eqs. (1) for $\nu=1$ have been studied by the similarity method in [19,20]. We have done a complete examination in order to realize whether the solutions presented there belong to any of the families of conjugate invariant solutions we have obtained. The basic conclusion from the comparison is that the families of invariant solutions (A1)–(A3), (A11), and (A12) include the solutions given in [19]. To support this observation we present in the Appendix B a list of several special assignments to the group parameters $a_1,\ldots,a_6$, allowing each of the solutions in [19] to be obtained from a certain family of invariant solutions for $\nu=1$. Moreover, wider classes of solutions of Eqs. (1) for $\nu=1$ are contained in Appendix A in comparison with [19], e.g., the solutions (A5) are not presented in [19,20]. As was already mentioned, in addition to [19,20], we analyze also the case $\nu=-1$.

Taken from [20], however, the solution

$$A=\sqrt{\frac{C_1}{x+b}}\exp i\left\{C_1\ln|x+b|+hC_2\ln|x+d|+\frac{t^2}{2(x+b)}+C_3\right\},$$
$$B=\sqrt{\frac{C_2}{x+d}}\exp i\left\{C_2\ln|x+d|+hC_1\ln|x+b|+\frac{t^2}{2(x+d)}+C_4\right\} \tag{25}$$

($C_1,C_2,C_3,C_4=\text{const}$; $b,d\in R$) is not among the invariant solutions considered in this paper. Moreover, it is easy to check the invariance of this solution in regard to a one-dimensional vector field

$$X=\frac{\partial}{\partial t}+\left(\frac{t}{x+b}\right)\frac{\partial}{\partial\alpha}+\left(\frac{t}{x+d}\right)\frac{\partial}{\partial\beta},$$

which does not belong to the Lie algebra based on the generators (4).

Finally, let us try to relate the results obtained here with those of [24]. The additional [in comparison to the system (1)] linear coupling in [24] allows one to describe switching phenomena in such systems [6]. The mathematical consequences from this are the lower dimension algebra of Lie, dimension 4, and correspondingly smaller classes of invariant solutions and laws of conservation. For example, in this case the full energy is conserved, but not the individual energy of each mode. Therefore, although some exact solutions may coincide, a direct comparison of both results is not possible.

VI. CONCLUSION

By means of the Lie group technique [21,22] we have studied a set of coupled nonlinear Schrödinger equations (1) describing nonlinear propagation in multimode optical fibers and fibers with strong birefringence. The most general Lie group of point symmetries, its Lie algebra, and the corresponding group of adjoint representations have been derived. Based on these, a complete list of group-invariant exact solutions has been obtained. The comparison with earlier results [19,20] reveals that the families of conjugate solutions obtained here include the solutions in [19,20] (except one in [20]) and a large number of others in addition. Therefore, the capabilities of the similarity approach used in [19] and the Lie group technique [21,22] are closely related to each other.

An exact solution of Eqs. (1), namely, Eqs. (14), and the corresponding family of invariant solutions (A5), different from the solutions in [19,20], have been obtained. Note, however, that the solution (25) presented in [20] does not belong to the invariant solutions obtained here. Further, by means of Noether's theorem, corresponding to the Hamiltonian symmetries obtained, conservation laws of Eqs. (1) also have been derived.

In conclusion, the results obtained here present a group classification of exact solutions of Eqs. (1). The so-called symbiotic periodic and soliton solutions are included in this classification in a natural way. The exact solutions can be used for tests in numerical solutions of Eqs. (1) and as trial functions for application of variational approach [30] in the analysis of different perturbed versions of Eqs. (1). Laws of conservation also can be used for the analysis of perturbed versions of Eqs. (1) and for the stability analysis of exact solutions (see [29]).

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APPENDIX A

Given in its entirety, each of the families of invariant solutions of Eqs. (1) comprises solutions, that depend on six arbitrary group parameters $a_1,\ldots,a_6$. In the list below the families are unified, for brevity, into five specific cases $A-E$, using for that purpose two additional parameters ε and δ. By setting various possible values for ε and δ one obtains different families of conjugate solutions. Other notations are C_1,C_2 arbitrary real constants and $p(\ ),q(\ ),f(\ ),g(\ )$ realvalued functions, which satisfy the respective reduce systems of equations given in Sec. III.

For case A,

$$A=e^{a_6}C_1\exp i\Bigg\{e^{a_6}(\varepsilon+a_5)t+e^{2a_6}\times\left(C_1^2+hC_2^2-\frac{\varepsilon^2+a_5^2}{2}-\varepsilon a_5\right)x-e^{2a_6}a_2\times\left(C_1^2+hC_2^2-\frac{\varepsilon^2+a_5^2}{2}-\varepsilon a_5\right)-e^{a_6}a_1(\varepsilon+a_5)+a_3\Bigg\},$$

$$B=e^{a_6}C_2\exp i\Bigg\{\nu e^{a_6}a_5t+e^{2a_6}\left(C_2^2+hC_1^2-\frac{\nu}{2}a_5^2\right)x-e^{2a_6}a_2\left(C_2^2+hC_1^2-\frac{\nu}{2}a_5^2\right)-\nu a_1a_5e^{a_6}+a_4\Bigg\},$$

$$(\varepsilon=0,\pm1). \quad \text{(A1)}$$

For case B,

$$A=\sqrt{\frac{C_1}{x-a_2}}\exp i\Bigg\{(C_1+hC_2)\ln|x-a_2|+\frac{(t-a_1)^2}{2(x-a_2)}+2a_6(C_1+hC_2)+a_3\Bigg\},$$

$$B=\sqrt{\frac{C_2}{x-a_2}}\exp i\Bigg\{(C_2+hC_1)\ln|x-a_2|+\nu\frac{(t-a_1+\nu\varepsilon e^{-a_6})^2}{2(x-a_2)}+2a_6(C_2+hC_1)-\varepsilon a_5+a_4\Bigg\}$$

$$(\varepsilon=0,\pm1). \quad \text{(A2)}$$

For case C,

$$A=e^{a_6}p(t')\exp i\Bigg\{f(t')+e^{a_6}a_5t+e^{2a_6}\left(\delta-\frac{a_5^2}{2}\right)x+\frac{1}{2}e^{2a_6}a_2a_5^2-a_1e^{a_6}a_5-\delta e^{2a_6}a_2+a_3\Bigg\},$$

$$B=e^{a_6}q(t')\exp i\Bigg\{g(t')+\nu e^{a_6}a_5t+e^{2a_6}\left(\varepsilon-\nu\frac{a_5^2}{2}\right)x+\frac{\nu}{2}e^{2a_6}a_2a_5^2-\nu a_1e^{a_6}a_5-\varepsilon e^{2a_6}a_2+a_4\Bigg\},$$

$$t'=e^{a_6}(t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1),$$

$$(\varepsilon=0,\ \delta=0,\pm1 \quad \text{or}\ \varepsilon=\pm1,\ \delta\in R). \quad \text{(A3)}$$

Having the form of Eqs. (A3), the next seven sets of solutions consist of uniform wave trains (A4), a special kind of knoidal wave (A5), solitons (A6), NPWs (A7), SPWs (A8), normal solitons (A9), and symbiotic solitons (A10). For case $C1$,

$$A=e^{a_6}C_1\exp i\Bigg\{e^{a_6}\left[a_5+\sigma\sqrt{2(C_1^2+hC_2^2-\delta)}\right]t-e^{2a_6}\left(\sigma a_5\sqrt{2(C_1^2+hC_2^2-\delta)}+\frac{a_5^2}{2}-\delta\right)x+\sigma\sqrt{2(C_1^2+hC_2^2-\delta)}\ e^{a_6}(e^{a_6}a_5a_2-a_1)-\delta a_2e^{2a_6}+\frac{1}{2}e^{2a_6}a_2a_5^2-a_1e^{a_6}a_5+a_3\Bigg\},$$

$$B=e^{a_6}C_2\exp i\Bigg\{e^{a_6}\left[\sigma\sqrt{\nu2(C_2^2+hC_1^2-\varepsilon)}+\nu a_5\right]t-e^{2a_6}\left(\sigma a_5\sqrt{\nu2(C_2^2+hC_1^2-\varepsilon)}+\nu\frac{a_5^2}{2}-\varepsilon\right)x+\sigma\sqrt{\nu2(C_2^2+hC_1^2-\varepsilon)}\ e^{a_6}(e^{a_6}a_5a_2-a_1)-\varepsilon a_2e^{2a_6}+\frac{\nu}{2}e^{2a_6}a_2a_5^2-\nu a_1e^{a_6}a_5+a_4\Bigg\}$$

$$(\varepsilon=0,\ \delta=0,\pm1 \quad \text{or}\ \varepsilon=\pm1,\ \delta\in R;\ \sigma=\pm1). \quad \text{(A4)}$$

For case $C2$, there are three families of invariant solutions, possible only in one region of GVD ($\nu=1$), which are written for case C, $\delta=\varepsilon$:

$$A=U\ \exp\ i\left\{\frac{C_1}{2\lambda\sqrt{h+1}\ b_1}\Pi(n;j|m)+\varphi-a_1a_5e^{a_6}+a_3\right\},$$

$$B=U\ \exp\ i\left\{\frac{\pm C_1}{2\lambda\sqrt{h+1}\ b_1}\Pi(n;j|m)+\varphi-a_1a_5e^{a_6}+a_4\right\},$$

$$U=e^{a_6}\sqrt{(b_1-b_2)\mathrm{cn}^2(j|m)+b_2},$$

$$\varphi=e^{a_6}a_5t+e^{2a_6}\left(\varepsilon-\frac{a_5^2}{2}\right)x-a_2\left(\varepsilon-\frac{a_5^2}{2}\right)e^{2a_6},$$

$$j=2\lambda\sqrt{h+1}(t-e^{a_6}a_5x+a_2a_5e^{a_6}-a_1)e^{a_6}$$

$$(\varepsilon=0,\pm1), \quad \text{(A5)}$$

[λ, m, and n are evaluated as in Eqs. (14).] For case $C3$ there is a family of invariant solitons of case C type for $\delta=\varepsilon=1$, which may exist only in one region of GVD ($\nu=1$):

$$A=e^{a_6}\sqrt{\frac{2}{h+1}}\ \mathrm{sech}\ j\ \exp\ i(\varphi-a_1a_5e^{a_6}+a_3),$$

$$B=e^{a_6}\sqrt{\frac{2}{h+1}}\ \mathrm{sech}\ j\ \exp\ i(\varphi-a_1a_5e^{a_6}+a_4), \quad \text{(A6)}$$

$$\varphi=e^{a_6}a_5t+e^{2a_6}\left(1-\frac{a_5^2}{2}\right)x-a_2\left(1-\frac{a_5^2}{2}\right)e^{2a_6},$$

$$j=\sqrt{2}(t-e^{a_6}a_5x+a_2a_5e^{a_6}-a_1)e^{a_6}.$$

Cases $C4$–$C7$ are valid only for two regions of GVD ($\nu=-1$):

$$t'=e^{a_6}(t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1),$$

$$\alpha'=e^{a_6}a_5t-e^{2a_6}\left(\frac{a_5^2}{2}-\delta\right)x+e^{2a_6}a_2\left(\frac{a_5^2}{2}-\delta\right)-a_1e^{a_6}a_5+a_3,$$

$$\beta'=-e^{a_6}a_5t+e^{2a_6}\left(\frac{a_5^2}{2}+\varepsilon\right)x-e^{2a_6}a_2\left(\frac{a_5^2}{2}+\varepsilon\right)+a_1e^{a_6}a_5+a_4.$$

For case $C4$,

$$A=e^{a_6}\sqrt{\frac{2\varepsilon}{m(1-h)+h+1}}\,\mathrm{dn}(jt'|m)\exp(i\alpha'),$$

$$B=e^{a_6}\sqrt{\frac{2\varepsilon m}{m(1-h)+h+1}}\,\mathrm{sn}(jt'|m)\exp(i\beta') \qquad (A7)$$

$$(\varepsilon=\pm1,\ \delta\in R,\ h<1).$$

For case $C5$,

$$A=e^{a_6}\sqrt{\frac{2(h+1-2\delta)}{(h-1)(h+3)}}\,\mathrm{sn}(jt'|m)\exp(i\alpha'), \qquad (A8)$$

$$B=e^{a_6}\sqrt{\frac{2(\delta+1)}{h+3}}\,\mathrm{dn}(jt'|m)\exp(i\beta')$$

$$\left(\varepsilon=1,\ \frac{2}{h+1}<\delta<\frac{h+1}{2},\ h>1\right).$$

For case $C6$,

$$A=e^{a_6}\,\mathrm{sech}(\sqrt{1-h}t')\exp(i\alpha'),$$

$$B=e^{a_6}\,\tanh(\sqrt{1-h}t')\exp(i\beta'), \qquad (A9)$$

$$\left(\varepsilon=1,\ \delta=\frac{h+1}{2},\ h<1\right).$$

For case $C7$,

$$A=e^{a_6}\sqrt{\frac{2}{h+1}}\,\tanh\left(\sqrt{\frac{2(h-1)}{h+1}}\,t'\right)\exp(i\alpha'),$$

$$B=e^{a_6}\sqrt{\frac{2}{h+1}}\,\mathrm{sech}\left(\sqrt{\frac{2(h-1)}{h+1}}\,t'\right)\exp(i\beta') \qquad (A10)$$

$$\times\left(\varepsilon=1,\ \delta=\frac{2}{h+1},\ h>1\right).$$

[j and m are evaluated as in Eqs. (16) and (17), respectively.] For case D,

$$A=e^{a_6}p(y)\exp i\left\{f(y)+\varepsilon e^{3a_6}tx-(\varepsilon a_2e^{2a_6}-a_5)e^{a_6}t-\frac{1}{3}e^{6a_6}x^3+e^{4a_6}(a_2e^{2a_6}-\varepsilon a_5)x^2+\left(2\varepsilon e^{2a_6}a_2a_5-\varepsilon a_1e^{a_6}-a_2^2e^{4a_6}-\frac{a_5^2}{2}\right)e^{2a_6}x-\varepsilon a_2(a_2a_5e^{a_6}-a_1)e^{3a_6}+\frac{1}{2}a_2a_5^2e^{2a_6}-a_1a_5e^{a_6}+\frac{1}{3}a_2^3e^{6a_6}+a_3\right\}, \qquad (A11)$$

$$B=e^{a_6}q(y)\exp i\left\{g(y)+\nu\varepsilon e^{3a_6}tx-\nu(\varepsilon a_2e^{2a_6}-a_5)e^{a_6}t-\frac{\nu}{3}e^{6a_6}x^3+\nu e^{4a_6}(a_2e^{2a_6}-\varepsilon a_5)x^2+\nu\left(2\varepsilon e^{2a_6}a_2a_5-\varepsilon a_1e^{a_6}-a_2^2e^{4a_6}-\frac{a_5^2}{2}+\nu\varepsilon\delta\right)e^{2a_6}x-\nu\varepsilon a_2\times(a_2a_5e^{a_6}-a_1)e^{3a_6}-\left(\varepsilon\delta-\nu\frac{{a_5}^2}{2}\right)a_2e^{2a_6}-\nu a_1a_5e^{a_6}-\frac{\nu}{3}a_2^3e^{6a_6}+a_4\right\},$$

$$y=e^{a_6}t-\frac{\varepsilon}{2}e^{4a_6}x^2-(a_5-\varepsilon a_2e^{2a_6})e^{2a_6}x+a_2a_5e^{2a_6}-a_1e^{a_6}-\frac{\varepsilon}{2}a_2^2e^{4a_6}, \quad (\varepsilon=\pm1,\ \delta\in R).$$

For case E,

$$A=\frac{p(y)}{t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1}\exp i\left\{f(y)-\varepsilon\ \ln|t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1|+e^{a_6}a_5t-\frac{1}{2}e^{2a_6}a_5^2x+\frac{1}{2}e^{2a_6}a_2a_5^2-a_1e^{a_6}a_5-\varepsilon a_6+a_3\right\}, \qquad (A12)$$

$$B=\frac{q(y)}{t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1}\exp i\left\{g(y)-\delta\ \ln|t-e^{a_6}a_5x+e^{a_6}a_5a_2-a_1|+\nu e^{a_6}a_5t-\frac{\nu}{2}e^{2a_6}a_5^2x+\frac{\nu}{2}e^{2a_6}a_2a_5^2-\nu a_1e^{a_6}a_5-\delta a_6+a_4\right\},$$

$$y=\frac{(t-e^{a_6}a_5x+e^{a_6}a_2a_5-a_1)^2}{x-a_2} \quad (\varepsilon,\delta\in R).$$

APPENDIX B

Here we give the assignments to the group parameters, allowing each of the solutions in [19] to be obtained from a certain family of invariant solutions. For Eqs. (A3) ($\nu=1$) implying case I in [19],

$$a_1=a_2=0,\quad a_3,a_4\in R,\quad \sqrt{2}e^{a_6}a_5=-V,$$

$$e^{2a_6}\left(\frac{a_5^2}{2}+\delta\right)=\gamma_1,\quad e^{2a_6}\left(\frac{a_5^2}{2}+\varepsilon\right)=\gamma_2$$

$$(\varepsilon=0,\ \delta=0,\pm 1\quad \text{or}\quad \varepsilon=\pm 1,\ \delta\in R).$$

For Eqs. (A2) ($\nu=1$) implying case IIa in [19],

$$a_1=-\frac{\gamma_1}{\sqrt{2}},\quad a_2=a_5=0,$$

$$a_3=-\varphi_1-2a_6(C_1+hC_2),$$

$$a_4=-\varphi_2-2a_6(C_2+hC_1),$$

$$(\gamma_2-\gamma_1)e^{a_6}=\varepsilon\sqrt{2}$$

$$(x<0,\ \varepsilon=0,\pm 1).$$

For Eqs. (A1) ($\nu=1$) implying case IIb in [19],

$$a_1=a_2=0,\quad a_3=\varphi_1,\quad a_4=\varphi_2,$$

$$(a_5+\varepsilon)e^{a_6}=-\gamma_1\sqrt{2},\quad a_5e^{a_6}=-\gamma_2\sqrt{2}$$

$$(\varepsilon=0,\pm 1).$$

For Eqs. (A11) ($\nu=1$) implying case III in [19],

$$a_1=-\frac{\gamma_1}{\sqrt{2}V},\quad a_2=a_5=0, a_6=\ln\sqrt[3]{\varepsilon V\sqrt{2}}$$

$$\left(a_3,a_4\in R;\quad \delta=\frac{\varepsilon(\gamma_2-\gamma_1)}{\sqrt[3]{2V^2}},\ \varepsilon=\pm 1\right).$$

For Eqs. (A12) ($\nu=1$) implying case IV in [19],

$$a_1=a_2=a_6=0,\quad (a_3,a_4\in R),$$

$$a_5=-\frac{V}{\sqrt{2}},\quad (\varepsilon=-2\gamma_1,\quad \delta=-2\gamma_2,\quad x<0).$$

For Eqs. (A12) ($\nu=1$) implying case V in [19],

$$a_1=a_2=a_5=0,\quad (a_3,a_4,a_6\in R),$$

$$(\varepsilon=-2\gamma_1,\quad \delta=-2\gamma_2,\quad x<0).$$

For Eqs. (A6) implying the soliton case in [19] (p. 276),

$$a_1=a_2=a_3=a_4=0,$$

$$e^{a_6}=\sqrt{\gamma_1-\frac{V^2}{4}},\quad a_5=-\frac{V}{\sqrt{2\left(\gamma_1-\frac{V^2}{4}\right)}}.$$

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